AR(1) $y_t = \mu + \phi_1 y_{t-1} + u_t$

$$= \mu + \phi_1 \cdot (\mu + \phi_1 y_{t-2} + u_{t-1}) + u_t$$
$$= \mu + \phi_1 \cdot \mu + \phi_1^2 \underline{y_{t-2}} + \phi_1 u_{t-1} + u_t$$
$$= \mu + \phi_1 \mu + \phi_1^2 (\mu + \phi_1 y_{t-3} + u_{t-2}) + \phi_1 u_{t-1} + u_t$$
$$= \mu + \phi_1 \mu + \phi_1^2 \mu + \phi_1^3 y_{t-3} + \phi_1^2 u_{t-2} + \phi_1 u_{t-1} + u_t$$
$$\vdots$$
$$= \mu + \phi_1 \mu + \phi_1^2 \mu + \dots + \phi_1^n \cdot \mu + \phi_1^{n+1} y_{t-(n+1)} + \phi_1^n u_{t-n} + \phi_1^{n-1} u_{t-(n-1)} + \dots + u_t$$
$$\vdots$$
$$= \mu (1 + \phi_1 + \phi_1^2 + \dots + \phi_1^n + \dots) + \underbrace{u_t + \phi_1 u_{t-1} + \phi_1^2 u_{t-2} + \dots}_{MA(\infty)}$$

$$E[y_t] = \mu (1 + \phi_1 + \phi_1^2 + \dots + \phi_1^n + \dots) + (1 + \phi_1 + \phi_1^2 + \dots) \cdot \underbrace{E[u_t]}_{0}$$
$$= \mu \cdot \frac{1}{1-\phi_1} + \frac{1}{1-\phi_1} \cdot 0$$
$$= \frac{\mu}{1-\phi_1}$$

AR(1) $y_t = \phi_1 y_{t-1} + u_t$

$\mu = 0$

$$var(y_t) = E[(y_t - E(y_t))^2] = E(y_t^2)$$
$$= E[(u_t + \phi_1 u_{t-1} + \phi_1^2 u_{t-2} + \phi_1^3 u_{t-3} + \dots)^2]$$
$$= E[u_t^2 + (\phi_1 u_{t-1})^2 + (\phi_1^2 u_{t-2})^2 + \dots$$
$$+ 2 u_t \cdot \phi_1 u_{t-1} + 2 u_t \phi_1^2 u_{t-2} + \dots]$$
$$= \sigma^2 + \phi_1^2 \sigma^2 + \phi_1^4 \sigma^2 + \dots + \phi_1^{2n} \sigma^2 + \dots$$
$$+ 2\phi_1 \cdot 0 + 2\phi_1^2 \cdot 0 + \dots$$
$$= \sigma^2 (1 + \phi_1^2 + \phi_1^4 + \dots + \phi_1^{2n} + \dots) = \sigma^2 \cdot \frac{1}{1-\phi_1^2}$$

$|\phi_1| < 1$

$$cov(y_t, y_{t-1}) = E[(y_t - E(y_t)) \cdot (y_{t-1} - E(y_{t-1}))]$$
$$= E[y_t \cdot y_{t-1}]$$
$$= E[(u_t + \phi_1 u_{t-1} + \phi_1^2 u_{t-2} + \dots) \cdot (u_{t-1} + \phi_1 u_{t-2} + \phi_1^2 u_{t-3} + \dots)]$$
$$= E[u_t u_{t-1} + \phi_1 u_{t-1}^2 + \phi_1^3 u_{t-2} \cdot u_{t-1} + \dots + \phi_1 u_t u_{t-2} + \phi_1^2 u_t u_{t-3} + \dots + \dots]$$
$$= \phi_1 \cdot E[u_{t-1}^2] + \phi_1^2 \cdot \phi_1 E[u_{t-2}^2] + \phi_1^2 \phi_1 \dots$$
$$= \phi_1 \cdot \sigma^2 + \phi_1^3 \cdot \sigma^2 + \phi_1^5 \sigma^2 + \dots$$
$$= \sigma^2 \phi_1 [1 + \phi^2 + \phi_1^4 + \dots]$$
$$= \sigma^2 \cdot \phi_1 \cdot \frac{1}{1-\phi_1^2}$$

$$var(y_t) = \sigma^2 \cdot \frac{1}{1-\phi_1^2}$$

$$Corr(y_t, y_{t-1}) = \frac{\sigma^2 \frac{\phi_1}{1-\phi_1^2}}{\sigma^2 \cdot \frac{1}{1-\phi_1^2}} = \phi_1$$